

Finite Math - J-term 2019  
Lecture Notes - 1/28/2019

## HOMEWORK

- Section 7.4 - 7, 9, 11, 13, 15, 17, 19, 21, 31, 32, 33, 34, 37, 38, 40, 41, 43, 51
- Section 8.1 - 1, 3, 5, 7, 9, 11, 15, 17, 18, 19, 25, 26, 33, 43, 45, 47, 49, 51, 53, 63, 65, 68

## SECTION 7.4 - PERMUTATIONS AND COMBINATIONS

**Example 1.** *From a committee of 12 people:*

- In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person cannot hold more than one position?*
- In how many ways can we choose a subcommittee of 4 people?*

**Example 2.** *Find the number of combinations of 30 objects taken 4 at a time.*

**Example 3.** *How many ways can a 3-person subcommittee be selected from a committee of 7 people? How many ways can a president, vice-president, and secretary be chosen from a committee of 7 people?*

**Example 4.** *Find the number of combinations of 67 objects taken 5 at a time.*

**Example 5.** *Suppose we have a standard 52-card deck and we are considering 5-card poker hands.*

- (a) *How many hands have 3 hearts and 2 spades?*
- (b) *How many hands have all the same suit? (I.e., what is the number of different flushes?)*
- (c) *How many possible pairs are there? (The other three cards have a different number from the pair and each other.)*
- (d) *How many possible 3 of a kinds are there? (The other two cards have a different number from the 3 of a kind and from each other.)*
- (e) *How many full houses are possible? (A full house consists of a three of a kind and a pair, each from a different number.)*

## SECTION 8.1 - SAMPLE SPACES, EVENTS, AND PROBABILITY

**Experiments.** If an experiment is repeated several times (such as rolling a die 100 times or flipping a coin 100 times), there is no guarantee, and in fact it is highly unlikely, that the same result will appear every time. This type of experiment is a *random experiment*. Probability is a field of mathematics dedicated to studying experiments with random outcomes.

### Sample Spaces and Events.

**Definition 1** (Sample Spaces and Events). *If we formulate a set  $S$  of outcomes of an experiment in such a way that in each trial of the experiment one and only one of the outcomes in the set will occur, we call the set  $S$  a sample space for the experiment. Each element in  $S$  is called a simple outcome, or simple event.*

*An event  $E$  is defined to be any subset of  $S$  (including the empty set  $\emptyset$  and the sample space  $S$ ). Event  $E$  is a simple event if it contains only one element and a compound event if it contains more than one element. We say that an event  $E$  occurs if any of the simple events in  $E$  occurs.*

There are some language differences when we are focused on probability in the real world versus theoretically:

| Real World | Theoretical  |
|------------|--------------|
| Experiment | Sample space |
| Outcome    | Event        |

In class, we will likely not make a distinction between the two, and probably even use them interchangeably.

**Example 6.** *Consider the number wheel with numbers 1-18 on it.*

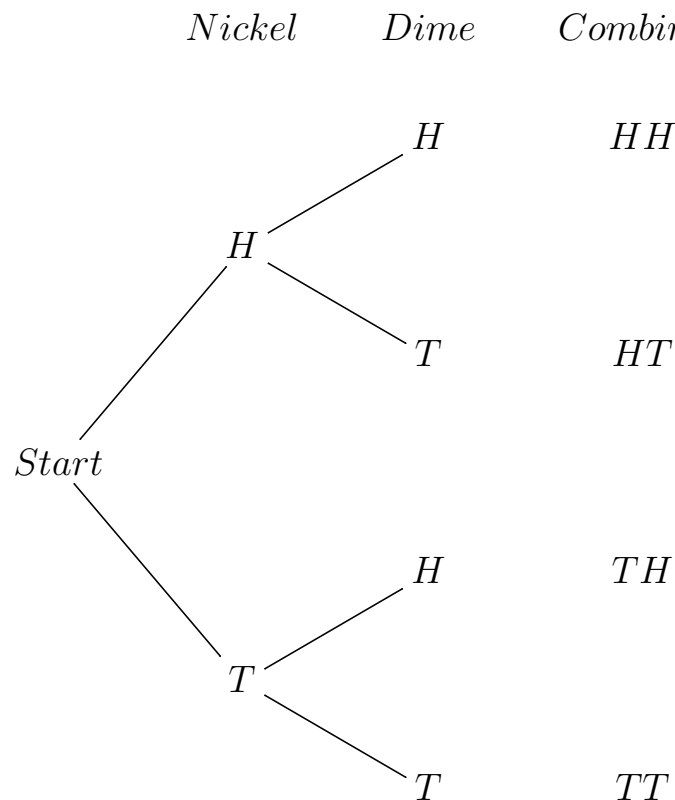
- (a) *What is the sample space?*
- (b) *What is the event corresponding to the outcome of the experiment being prime? Is it a simple event or a compound event?*
- (c) *What is the event corresponding to the outcome of the experiment being the square of 4? Is it a simple event or a compound event?*

**Example 7.** *Consider again the number wheel with numbers 1-15 on it.*

- (a) *What is the sample space?*
- (b) *What is the event corresponding to the outcome of the experiment being divisible by 12? Is it a simple event or a compound event?*
- (c) *What is the event corresponding to the outcome of the experiment being even? Is it a simple event or a compound event?*

The sample spaces can vary depending on what we're interested in.

**Example 8.** *A nickel and a dime are flipped. Depending on what we want to focus on, the sample space can vary. First, let's look at what the raw outcomes are*



- (a) *If we are interested in knowing whether each coin was heads or tails, an appropriate sample space is*

*since this tells us which coin came up which.*

- (b) *If we are only interested in the number of heads that comes up, we can choose*

*as our sample space since this describes how many heads come up.*

- (c) *Suppose we only want to know if the coins match or not. Then we can use*

*as a sample space where  $M$  stands for match and  $D$  stands for do not match.*

Notice that the sample space in (a) can actually be used in the other two situations as well, and for this reason we say that  $S_1$  is a more *fundamental sample space* than the other two.

**Example 9.** *An experiment consists of recording the boy-girl composition of a three-child family. What would be an appropriate sample space if:*

- (a) *we are interested in the genders of the children in the order of their births?  
(A tree diagram can help.)*
- (b) *we are interested only in the number of girls in family?*
- (c) *we are interested only in which gender there are more of?*
- (d) *we are interested in all three items from (a)-(c)?*

**Example 10.** Suppose we run an experiment by rolling two dice. What is the most fundamental sample space for this experiment? Give the event for each of the following outcomes. Which are simple events?

- (a) A sum of 7 turns up.
- (b) A sum of 11 turns up.
- (c) A sum less than 4 turns up.
- (d) A sum of 12 turns up.
- (e) A sum of 5 turns up.
- (f) A sum which is prime turns up.
- (g) A sum of 2 turns up.

## Probability of an Event.

**Definition 2** (Probabilities of Simple Events). *Given a sample space*

$$S = \{e_1, e_2, \dots, e_n\}$$

*with  $n$  simple events, to each simple event  $e_i$  we assign a real number, denoted by  $P(e_i)$ , called the probability of event  $e_i$ . These numbers can be assigned in an arbitrary manner as long as the following two conditions are satisfied:*

Condition 1: *The probability of a simple event is a number between 0 and 1, inclusive, i.e.,*

$$0 \leq P(e_i) \leq 1.$$

Condition 2: *The sum of the probabilities of all simple events in the sample space is 1, i.e.,*

$$P(e_1) + P(e_2) + \dots + P(e_n) = 1.$$

*Any probability assignment that satisfies Conditions 1 and 2 is said to be an acceptable probability assignment.*

A simple example to illustrate this is flipping one coin. The sample space for this would be  $S = \{H, T\}$ . We would assume that there's a "50-50 chance" that the coin will turn up heads or tails, so it seems reasonable to assign the following probabilities:

$$P(H) = \frac{1}{2} \quad P(T) = \frac{1}{2}.$$

Notice that this satisfies the two conditions of the previous definition. Is this a reasonable assignment of probabilities? Ostensibly yes since there are only two outcomes of a coin flip and there is no reason to doubt that the coin is *fair*, i.e., the two outcomes are equally likely.

However, suppose we flipped the coin 1000 times and we turn up with 373 heads and 627 tails. This could suggest that the coin is somehow "rigged" to land on tails more often. In this case, we might consider assigning the probabilities as

$$P(H) = \frac{373}{1000} = 0.373 \quad P(T) = \frac{627}{1000} = 0.627$$

since that reflects the actual results of a large collection of outcomes.

Another assignment we could technically make is

$$P(H) = 1 \quad P(T) = 0.$$

While this does fit the rules of an acceptable probability assignment, it is not *reasonable* in this case, unless the coin had two heads.

An example of an unacceptable probability assignment is

$$P(H) = 0.6 \quad P(T) = 0.8$$

since  $P(H) + P(T) = 1.4 > 1$ .

**Definition 3** (Probability of an Event  $E$ ). *Given an acceptable probability assignment for the simple events in a sample space  $S$ , we define the probability of an arbitrary event  $E$ , denoted  $P(E)$ , as follows:*

- (a) *If  $E$  is the empty set, then  $P(E) = 0$ .*
- (b) *If  $E$  is a simple event, then  $P(E)$  has already been assigned.*
- (c) *If  $E$  is a compound event, then  $P(E)$  is the sum of the probabilities of all the simple events in  $E$ .*
- (d) *If  $E$  is the sample space  $S$ , then  $P(E) = P(S) = 1$  (this is a special case of part (c).)*

**Example 11.** *Refer back to the example where we roll two dice. If we assume that every simple event is equally likely:*

- (a) *What is the probability of a simple event happening?*
- (b) *What are the possible numbers that the two dice could add up to?*
- (c) *What are the probability of each of the events in part (b) happening?*

**Empirical Approach.** In an empirical approach to probability, we run the experiment several times, and assign probabilities according to the frequency which with outcomes occurred. For example, if we flip a coin 1000 times and get 373 heads and 627 tails, we would be tempted to assign probabilities as

$$P(H) = \frac{373}{1000} \quad P(T) = \frac{627}{1000}$$

since it reflects the results of an extensive experiment.

The number of times an event  $E$  occurs in an experiment is called the *frequency* of the event, and is denoted  $f(E)$ . If the experiment has  $n$  trials (in the example above,  $n = 1000$  since there was 1000 coin flips), the *relative frequency* of the event  $E$  in  $n$  trials is the number  $\frac{f(E)}{n}$ . We can define the *empirical probability* of  $E$ ,

which we will denote by  $P(E)$ , by the number that  $\frac{f(E)}{n}$  approaches as  $n$  gets larger and larger. The reasoning behind this is that given an infinite number of runs of the experiment, the relative frequency should reflect the actual probabilities, and generally, this number becomes a better and better approximation as the number of runs of the experiment increases. For any finite number  $n$  we plug in,  $\frac{f(E)}{n}$  is an *approximate empirical probability* of the event  $E$ .

**Equally Likely Assumption.** When we were talking about assigning probabilities of 0.5 to heads and 0.5 to tails for flipping a coin, and a probability of  $\frac{1}{6}$  for any number to come up when rolling a 6-sided die, we are making an assumption on the probabilities of the experiment called an *equally likely assumption*. In an ideal case, this poses no risk, but as we talked about, a coin or a die may not necessarily be “fair.”

Generally, if the sample space is

$$S = \{e_1, e_2, \dots, e_n\},$$

we assign to each  $e_i$  a probability of  $\frac{1}{n}$  since there are  $n$  possible outcomes and we want each of them to be equally likely. This gives us the following theorem

**Theorem 1** (Probability of an Arbitrary Event under an Equally Likely Assumption). *If we assume that each simple event in a sample space  $S$  is equally likely to occur, then the probability of an arbitrary event  $E$  in  $S$  is given by*

$$P(E) = \frac{n(E)}{n(S)},$$

*the number of elements in  $E$  divided by the number of elements in  $S$ .*

We saw this theorem in action when we found the theoretical probabilities for rolling a number on a pair of dice.